UNIFORM CONVERGENCE AND THE FREE CENTRAL LIMIT THEOREM

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Abstract. We prove results about uniform convergence of densities in the free central limit theorem without assumptions of boundedness on the support.

1. Introduction

Consider free, identically distributed random variables $X_1, X_2, \dots$ such that $E(X_j) = 0$ and $E(X_j^2) = 1$. The free central limit theorem tells us that the distribution μ_n of the random variable $n^{-1/2}(X_1 + X_2 + \cdots + X_n)$ converges weakly to the semicircle law γ, given by

$$d\gamma(t) = (2\pi)^{-1}\sqrt{4-t^2}\chi_{(-2,2)}(t)dt.$$

It was shown in [2] that, when the variables X_j are bounded, μ_n is absolutely continuous for big enough n, and its density converges to $(2\pi)^{-1}\sqrt{4-x^2}$ uniformly.

It is our purpose to extend this result to unbounded random variables. In particular, we show that $d\mu_n/dt$ converges to $(2\pi)^{-1}\sqrt{4-x^2}$ uniformly on compact subsets of $(-2,2)$. When μ is infinitely divisible, we show that this convergence is uniform on $\mathbb{R}$. The main results have recently been superseded in [9] with a proof making stong use of free brownian motion.

2. Preliminaries.

A noncommutative probability space is defined to be a pair (A, φ), where A is a W^*-algebra and φ is a tracial state on A. A bounded random variable is an element $a \in A$. A family of unital subalgebras $\{A_i\}_{i\in I} \subseteq A$ is said to be freely independent if $\varphi(a_1a_2\cdots a_n) = 0$ whenever $a_j \in A_{i_j}$ with $i_j \neq i_{j+1}$ for $j = 1, 2, \dots, n-1$ and $\varphi(a_k) = 0$ for $k = 1, 2, \dots, n$.

Given a random variable $x \in A$, its distribution is the functional $\mu_x : \mathbb{C}[X] \to \mathbb{C}$ defined by the property that $\mu_x(P(X)) = \varphi(P(a))$ for $P \in \mathbb{C}[X]$. If $x = x^*$, then μ_x is given by integration against a compactly supported probability measure on the real line. More specifically, let E_x denote the spectral measure of x, and define a Borel measure μ on $\mathbb{R}$ by setting $\mu(\sigma) = \varphi(E_x(\sigma))$. Then we have $\varphi(P(x)) = \int_{\sigma(x)} p(t)d\mu(t)$ where $\sigma(x)$ denotes the spectrum of our random variable

Now, consider self-adjoint, freely independent random variables $x, y \in A$ and their corresponding probability distributions, μ_x and μ_y. It was established in [8] that μ_{x+y} depends only on μ_x and μ_y. As such, we can define the additive free convolution operation $\boxplus$ on the space Σ of all linear functionals on $\mathbb{C}[X]$ via the formula $\mu_x \boxplus \mu_y = \mu_{x+y}$. Finding $\mu_x \boxplus \mu_y$

given μ_x and μ_y is very nontrivial in practice and we introduce some of the analytic tools that come to bear on this problem.

Given a probability measure μ defined on $\mathbb{R}$, we define the Cauchy transform to be the analytic function, $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ where

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t}.$$

For positive real numbers α and β, let us set

$$\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ : \Im(z) > \alpha|\Re(z)|, |z| > \beta.\}.$$

Note that for fixed $\alpha > 0$, there exists a $\beta > 0$ so that $G_\mu(z)$ maps $\Gamma_{\alpha,\beta}$ injectively onto a region in $\mathbb{C}^-$ containing z^{-1} for all $z \in \Gamma_{\alpha',\beta'}$ with $\alpha > \alpha'$ and $\beta > \beta'$. We define the R-transform of μ by the formula $R_\mu(z) = G_\mu^{-1}(z) - z^{-1}$. This function may be defined for all $z \in \{z \in \mathbb{C}^- | -\Im(z) \geq \gamma|\Re(z)|, 0 < \Im(z) < \lambda\}$ where $\gamma, \lambda \in \mathbb{R}^+$ depend on our measure μ, and maps into a set $\Gamma_{\alpha,\beta}$ as was previously defined. The R-transform satisfies the property that $R_{\mu_x \boxplus \mu_y}(z) = R_{\mu_x}(z) + R_{\mu_y}(z)$ on an appropriate angle, as above, where all are defined [8]. A fundamental technique in finding $\mu_x \boxplus \mu_y$ when given μ_x and μ_y for x and y freely independent random variables is to compute the R-transform of each and then find the probability measure whose R-transform corresponds to their sum.

It turns out that it is advantageous to consider the reciprocal of the Cauchy transform, $F_\mu(z) = G_\mu^{-1}(z)$. We define the function that corresponds to the R-transform in this setting, $\varphi_\mu(z) := F_\mu^{-1}(z) - z$. We also define $\mathbb{C}_a^+ = \{z \in \mathbb{C}^+ : \Im(z) > a\}$ where $a \in \mathbb{R}^+$. The following properties were established in [6] and will be used throughout this paper.

Lemma 2.1. *For any probability measure μ defined on $\mathbb{R}$ having finite variance, $|F_\mu(z) - z| \leq C/\Im(z)$ for $z \in \mathbb{C}^+$ where $C > 0$ depends only on μ.*

Lemma 2.2. *For measure μ with zero mean and variance σ^2, $F_\mu^{-1}(z) : \mathbb{C}_{2\sigma}^+ \to \mathbb{C}_\sigma^+$ is defined and satisfies $|\varphi_\mu(z)| = |F_\mu^{-1}(z) - z| \leq 2\sigma^2/\Im(z)$ for $z \in \mathbb{C}_{2\sigma}^+$*

Lemma 2.3. *For μ a probability measure on $\mathbb{R}$ and $z \in \mathbb{C}^+$, we have that $\Im F_\mu(z) \geq \Im z$ with equality for some $z \in \mathbb{C}^+$ if and only if μ is a Dirac measure.*

We also have the following property, first established for the R-transform in [8] and which is easily seen to hold for φ_μ.

Lemma 2.4. *For μ and ν probability measures on $\mathbb{R}$, $\varphi_{\mu\boxplus\nu}(z) = \varphi_\mu(z) + \varphi_\nu(z)$.*

A measure is said to be $\boxplus$-infinitely divisible if, for all $n \geq 2$, there exists a probably measure $\mu_{1/n}$ such that $\mu = \mu_{1/n}^{\boxplus n}$. These measures were introduced in [8] and have proven to be very well behaved with respect to free harmonic analysis. The following properties, proved in [1], exemplify this observation.

Lemma 2.5. *For μ an infinitely divisible measure on $\mathbb{R}$ with variance σ^2, the following properties hold:*

1. *$F_\mu(z) + \varphi_\mu(F_\mu(z)) = z$ for every $z \in \mathbb{C}^+$.*
2. *The inequality $|\varphi_\mu(z)| \leq 2\sigma^2/\Im(z)$ holds for all $z \in \mathbb{C}^+$.*

(3) *We have that*

$$\varphi_\mu(z) = \alpha + \int_{-\infty}^{\infty} \frac{1+tz}{z-t} d\nu(t)$$

for ν is a probability measure. Moreover, if μ is not a dirac measure, we have that $\nu([a,b]) > 0$ for some $a, b \in \mathbb{R}$ and $\alpha \in \mathbb{R}$.

Recall the measure γ defined by the density function $d\gamma(t) = (2\pi)^{-1}\sqrt{4-t^2}\chi_{(-2,2)}(t)dt$. We refer to this measure as the *semicircle law.* The semicircle law plays a role in free probability theory that is in many ways analogous to that of the Gaussian law in classical probability theory. Representative of this is the free central limit theorem, first proved in [7].

Theorem 2.6. *Let (A, φ) be a noncommutative probability space and $\{a_j\}_{j=1}^{\infty} \subseteq A$ be a free family of random variables such that:*

(1) $\varphi(a_j) = 0$ *for all* $j \leq 1$
(2) $\sup_{j\leq 1} |\varphi(a_j^k)| < \infty$ *for each* $k \leq 2$
(3) $\lim_{n\to\infty} n^{-1}\Sigma_{j=1}^{n} \varphi(a_j^2) = 1.$

Then $n^{-1/2}(a_1 + ... + a_n)$ converges in distribution to the semicircle law.

Consider noncommutative probability space (A, φ) and assume that A is acting on a Hilbert space H. A self-adjoint operator T acting on H is said to be affiliated with A (in symbols, $T\eta A$), if the spectral projections of T belong to A. We note that our definition of distribution, $\mu_T(\sigma) := \varphi(E_T(\sigma))$ extends to these operators since, by assumption, $E_T(\sigma) \in A$. These distributions will be probability measures on $\mathbb{R}$ whose support is not generally bounded. This is the class of measures we address in this paper.

3. Uniform convergence on compact subsets of $(-2, 2)$.

Denote by μ_n the distribution of $n^{-1/2}(X_1 + \cdots + X_n)$ where $X_1, \ldots, X_n$ are freely independent random variables with distribution μ.

The following lemmas are known and can be found, more or less explicitly, in [6]. Proofs are presented for the reader's convenience.

Lemma 3.1. *For any $\alpha > 0$ there exists $\beta > 0$ so that φ_{μ_n} is defined on $\Gamma_{\alpha,\beta}$ and converges uniformly to z^{-1} on compact subsets of $\Gamma_{\alpha,\beta}$.*

Proof. As seen in [6] φ_{μ_n} is defined on $\{z \in \mathbb{C}^+ : \Im(z) > 2\}$. Thus, for fixed $\alpha > 0$, we need only pick β big enough so that $\Gamma_{\alpha,\beta}$ lies in the above set.

To prove the uniform convergence result, pick $\Omega \subseteq \Gamma_{\alpha,\beta}$. Note that $\{\varphi_{\mu_n}\}$ is a normal family since Lemma 2.2 implies a uniform bound of 1 on all of $\Gamma_{\alpha,\beta}$. Therefore, we have subsequences which converge uniformly on compact subsets of $\Gamma_{\alpha,\beta}$ and we need only show that any such subsequence converges to z^{-1}. Let $\varphi_{\mu_{n_j}}$ converge to φ. For γ, the semicircle measure,

$$|F_\gamma(z + \varphi(z)) - z| = |F_\gamma(z + \varphi(z)) - F_{\mu_{n_j}}(\varphi_{\mu_{n_j}}(z) + z)|$$

$$\leq |F_\gamma(z + \varphi(z)) - F_\gamma(\varphi_{\mu_{n_j}}(z) + z)| + |F_\gamma(\varphi_{\mu_{n_j}}(z) + z) - F_{\mu_{n_j}}(\varphi_{\mu_{n_j}}(z) + z)|$$

By the free central limit theorem, $F_{\mu_{n_j}}$ converges to F_γ uniformly on a neighborhood of $z+\varphi(z)$, and this implies that $|F_\gamma(\varphi_{\mu_{n_j}}(z)+z)-F_{\mu_{n_j}}(\varphi_{\mu_{n_j}}(z)+z)|$ converges to 0. Therefore, $z+\varphi(z)=F_\gamma^{-1}(z)$ on $\Gamma_{\alpha,\beta}$ which tells us that $\varphi=\varphi_\gamma=z^{-1}$. □

Lemma 3.2. *For any $\alpha,\beta>0$ and n big enough, $\sqrt{n}\varphi_\mu(\sqrt{n}z)$ is defined for all $z\in\Gamma_{\alpha,\beta}$, agrees with φ_{μ_n} where both are defined and converges to z^{-1} uniformly on this set.*

Proof. As was proven in [6], φ_μ is defined on $\{z\in\mathbb{C}^+:\Im(z)>2\}$. Pick n large enough so that $\Im(\sqrt{n}z)>2$ for all $z\in\Gamma_{\alpha,\beta}$. Then $\sqrt{n}\varphi(\sqrt{n}z)$ is defined.

Now, by definition, $\varphi_{\mu_n}(F_{\mu_n}(z))+F_{\mu_n}(z)=z$ for all z with $\Im(z)>2$. Furthermore, it was established in [1] that $\varphi_{\mu_n}(z)=\sqrt{n}\varphi_\mu(\sqrt{n}z)$ on this set. Therefore, $\sqrt{n}\varphi_\mu(\sqrt{n}F_{\mu_n}(z))+F_{\mu_n}(z)$ is defined on all of $\Gamma_{\alpha,\beta}$ and is equal to the identity for those z with $\Im(z)>2$. By analytic continuation, this implies that $\sqrt{n}\varphi_\mu(\sqrt{n}F_{\mu_n}(z))+F_{\mu_n}(z)=z$ for all $z\in\Gamma_{\alpha,\beta}$. This implies, by definition of φ_{μ_n}, that the two functions agree for those $z\in\Gamma_{\alpha,\beta}$ where both are defined.

With regard to the question of uniform convergence on $\Gamma_{\alpha,\beta}$, choose α , β, ϵ >0 and $\Omega\subseteq\Gamma_{\alpha,\beta}$ compact. By Lemma 3.1, there exists a $\beta'>0$ so that φ_{μ_n} is defined on $\Gamma_{\alpha,\beta'}$ for all n and converges to z^{-1} uniformly on all compact subsets. Now, pick k such that $\sqrt{k}\Gamma_{\alpha,\beta}\subseteq\Gamma_{\alpha,\beta'}$. By the previous lemma, there exists $N>0$ such that for all $n\geq N$ we have that

$$|\varphi_{\mu_n}(\alpha z)-(\alpha z)^{-1}|<(k+1)^{-1/2}\epsilon$$

for all $\alpha\in[k,k+1]$ and $z\in\Omega$. We assume that $N>k$ and, for $n\geq N$, consider numbers of the form $\alpha=\sqrt{k+\ell n^{-1}}$ for $0\leq\ell<k$. We have that

$$|\varphi_{\mu_n}((\sqrt{k+\ell n^{-1}})z)-((\sqrt{k+\ell n^{-1}})z)^{-1}|<(k+1)^{-1/2}\epsilon\leq(k+\ell n^{-1})^{-1/2}\epsilon$$

$$\Rightarrow|\sqrt{k+\ell n^{-1}}\varphi_{\mu_n}((\sqrt{k+\ell n^{-1}})z)-z^{-1}|<\epsilon$$

and, since the previous lemma implies that $\varphi_{\mu_n}(z)=\sqrt{n}\varphi_\mu(\sqrt{n}z)$ for all $z\in\Gamma_{\alpha,\beta'}$, we have the following:

$$|\sqrt{nk+\ell}\varphi_\mu((\sqrt{nk+\ell})z)-z^{-1}|<\epsilon$$

for all $z\in\Omega$ and $0\leq\ell<k$. This implies that

$$|\sqrt{m}\varphi_\mu((\sqrt{m})z)-z^{-1}|<\epsilon$$

for all $m\geq Nk$. Thus, we have uniform convergence on compact subsets of $\Gamma_{\alpha,\beta}$

For uniform convergence on all of $\Gamma_{\alpha,\beta}$, by Lemma 2.2, $\varphi_{\mu_n}(z)$ goes to zero uniformly as $|z|\to\infty$ in $\Gamma_{\alpha,\beta}$. As the same holds for z^{-1}, by proving uniform convergence on compact sets, we have the general result as well. □

Theorem 3.3. *Let μ be a measure with mean 0 and variance 1. Then $d\mu_n/dt$ converges uniformly to the semicircle law on compact subsets of $(-2,2)$.*

Proof. It was shown in [3] that μ_n is absolutely continuous with respect to the Lebesgue measure for n large enough. We turn our attention to the question of convergence.

Consider the interval $[-2+\epsilon,2-\epsilon]$ and let $\Lambda_1=\{z:|z|=1,\Im(z)\geq\delta\}$ where δ is chosen so that $\{z+z^{-1}:z\in\Lambda_1\}=[-2+\epsilon,2-\epsilon]$. Let $\Lambda_2=\{\lambda i:1<\lambda<3\}$. We denote by Ω a $2^{-1}\delta$ neighborhood of $\Lambda_1\cup\Lambda_2$.

Note that $\Omega \subset \Gamma_{\alpha,\beta}$ for appropriate α and β. Observe that $z + z^{-1}$ maps Ω conformally onto a neighborhood of $[-2+\epsilon, 2-\epsilon] \cup i[0,2]$. By lemma 3.2 and Rouche's theorem, the same must hold for $\gamma_n(z) = \sqrt{n}\varphi_\mu(\sqrt{n}z) + z$ when n is large enough. Thus, there exists a partition $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ with the following properties:

(1) Ω_1 and Ω_2 are open and connected.
(2) $\gamma_n(\Omega) \cap \mathbb{C}^+ = \gamma_n(\Omega_1)$.
(3) $\gamma_n(\Omega) \cap \mathbb{C}^- = \gamma_n(\Omega_2)$.
(4) $\gamma_n(\Omega) \cap \mathbb{R} = \gamma_n(\Omega_3)$.
(5) $2i \in \Omega_1$.

Now, pick $t \in [-2+\epsilon, 2-\epsilon]$. There exists a positive real number h_t and a path $z^n_t : [0, h_t] \to \Omega$ so that $\gamma_n(z^n_t(h)) = t + ih$. Note that $z^n_t(h) \in \Omega_1$ for all $h \in [0, h_t]$. Lemmas 3.1 and 3.2 imply that $F_{\mu_n}(\gamma_n(z)) = z$ for those $z \in \Omega_1$ with $\Im(z) > 2$. As Ω_1 is open and connected, this must hold on the entire set by analytic continuation. Thus,

$$F_{\mu_n}(t+ih) = F_{\mu_n}(\gamma_n(z^n_t(h))) = z^n_t(h).$$

Since γ_n converges to $z+z^{-1}$ uniformly on Ω, we have that $z^n_t(0) \to (2^{-1}t+i(\sqrt{1-(2^{-1}t)^2}))$ uniformly over $t \in [-2+\epsilon, 2-\epsilon]$ and $n \to \infty$.

Now, by the Stieltjes inversion formula,

$$\frac{d\mu_n}{dx}(t) = \lim_{h\downarrow 0} \frac{-1}{\pi}\Im(G_{\mu_n}(t+ih)) = \lim_{h\downarrow 0} \frac{-1}{\pi}\Im(F_{\mu_n}(t+ih)^{-1})$$
$$= \lim_{h\downarrow 0} \frac{-1}{\pi}\Im(z^t_n(h)^{-1}) = \frac{-1}{\pi}\Im(z^t_n(0)^{-1}) \to \frac{-1}{\pi}\Im(2^{-1}(t-\sqrt{t^2-4})) = \frac{1}{2\pi}\sqrt{4-t^2}$$

As the convergence $z^n_t(0) \to ((2^{-1}t)^2 + i(\sqrt{1-(2^{-1}t)^2}))$ is uniform over $t \in [-2+\epsilon, 2-\epsilon]$, we have that $\frac{d\mu_n}{dx}(t) \to \frac{1}{2\pi}\sqrt{4-t^2}$ uniformly over $t \in [-2+\epsilon, 2-\epsilon]$, completing the proof. □

For any probability measure μ, define the cumulative distribution function $f_\mu(t) := \mu((-\infty, t])$. A number of free analogues of Berry-Esseen have been proven with respect to this function. Most notably, in [5], it was shown that for μ a measure with bounded support,

$$|f_{\mu_n}(t) - f_\gamma(t)| \leq CL^3 n^{-1/2}$$

where C is an absolute constant and $\mathrm{supp}(\mu) \subseteq [-L, L]$. It was shown in [4] that

$$|f_{\mu_n}(t) - f_\gamma(t)| \leq C(|m_3(\mu)| + m_4(\mu)^{1/2})n^{-1/2}$$

where $m_k(\mu)$ denotes the *kth* moment of the measure μ and C is and absolute constant. From Theorem 3.3 we derive the following partial result, stronger insofar as it makes no moment assumptions beyond the second and does not require compact support of the measure, but weaker since it does not provide a definite rate of convergence with respect to n:

Corollary 3.4. *For μ a probability measure with mean 0 and variance 1, $|f_{\mu_n}(t) - f_\gamma(t)| \to 0$ uniformly as $n \to \infty$.*

Proof. Choose $\epsilon > 0$. Pick $\delta > 0$ such that $\gamma([-2+\delta, 2-\delta]) > 1 - 4^{-1}\epsilon$. By Theorem 3.3, there exists an N such that for all $n \geq N$, $|d\mu_n/dt(x) - d\gamma/dt(x)| < 8^{-1}\epsilon$ for all $x \in [-2+\delta, 2-\delta]$. For $\sigma \subseteq [-2+\delta, 2-\delta]$, we have the following:

$$|\mu_n(\sigma) - \gamma(\sigma)| \leq \int_\sigma |\frac{d_\gamma}{dt}(x) - d_{\mu_n}/dt(x)|dt \leq 8^{-1}\epsilon|\sigma| \leq 2^{-1}\epsilon$$

Bearing in mind that μ_n is a probability measure, we have the following:

(1) For $t \leq -2+\delta$, we have that $f_{\mu_n}(t) = \mu_n((-\infty, t])) = 1 - \mu_n((t, \infty)) \leq 1 - \mu_n([-2+\delta, 2-\delta]) \leq 1 - (1 - \epsilon/2) = \epsilon/2$.

(2) For $t \in [-2+\delta, 2-\delta]$, we have that $f_{\mu_n}(t) = \mu_n((-\infty, -2+\delta)) + \mu_n([-2+\delta, t]) \leq \epsilon/2 + \epsilon/2$.

(3) For $t > 2 - \delta$, we have that $f_{\mu_n}(t) \geq \mu_n([-2+\delta, 2-\delta]) > 1 - \epsilon$.

Thus, our claim holds. ∎

It would be interesting to see whether $|f_{\mu_n}(t) - f_\gamma(t)| \leq Cn^{-1/2}$, for C an absolute constant, with no assumptions on the boundedness of the support and no assumptions on the existence of moments beyond the second.

4. Uniform convergence for infinitely divisible measures.

We begin with a lemma before proving the main result of the section.

Lemma 4.1. *Let μ be an infinitely divisible measure with mean 0 and variance 1. Then there exists a $C > 0$ so that for n big enough,*

$$F_{\mu_n}(\mathbb{C}^+) \subseteq \{z \in \mathbb{C}^+ : |z| > C\}$$

Proof. By Lemma 2.5 (2), we see that

$$\Im(\varphi_{\mu_n}(z)) = \Im(\sqrt{n}\varphi_\mu(\sqrt{n}z)) = \sqrt{n}\Im\int_{-\infty}^{\infty} \frac{1+\sqrt{n}tz}{\sqrt{n}z - t} d\nu(t) = -\int_{-\infty}^{\infty} \frac{y(1+t^2)}{(x - n^{-1/2}t)^2 + y^2} d\nu(t)$$

where $z = x + iy$. Now for $\nu([a, b]) > 0$, we have the following:

$$-\Im(\varphi_{\mu_n}(z)) = \int_{-\infty}^{\infty} \frac{y(1+t^2)}{(x - n^{-1/2}t)^2 + y^2} d\nu(t) \geq \int_a^b \frac{y(1+t^2)}{(x - n^{-1/2}t)^2 + y^2} d\nu(t)$$

$$\geq \frac{y(\nu([a, b]))}{\max\{|z - n^{-1/2}|b||^2, |z - n^{-1/2}|a||^2\}}$$

Now, for $z \in \mathbb{C}^+$, by Lemma 2.5 (1), $F_{\mu_n}(z) + \varphi_{\mu_n}(F_{\mu_n}(z)) = z$ which implies that $\Im(\varphi_{\mu_n}(F_{\mu_n}(z))) + \Im(F_{\mu_n}(z)) > 0$. Therefore, $-\Im(\varphi_{\mu_n}(F_{\mu_n}(z))) < \Im(F_{\mu_n}(z))$ and, setting $F_{\mu_n}(z) = x' + iy'$, we have that:

$$\frac{y'\nu([a, b])}{\max\{|F_{\mu_n}(z) - |z - n^{-1/2}|b||^2, |F_{\mu_n}(z) - |z - n^{-1/2}|b||^2\}} \leq y'$$

$\Rightarrow \nu([a, b]) \leq \max\{|F_{\mu_n}(z) - n^{-1/2}|b|\ |^2, |F_{\mu_n}(z) - n^{-1/2}|a|\ |^2\}$. As $n^{-1/2}|a|\ , n^{-1/2}|b| \to 0$, we get that for n big enough, $C = \sqrt{\nu([a, b])/2} \leq |F_{\mu_n}(z)|$, which proves our lemma. ∎

We now formulate and prove our main theorem.

Theorem 4.2. *Let μ be an infinitely divisible measure with mean 0 and variance 1. Then $\frac{d\mu_n}{dt}$ converges to the semicircle law uniformly.*

Proof. We already know from Theorem 3.3 that $\lim_{n\to\infty} d\mu_n(x) = (2\pi)^{-1}\sqrt{4-x^2}$ uniformly on compact subintervals of $(-2,2)$. Assuming, for the sake of contradiction, that we do not have uniform convergence of the density, we get a sequence of real numbers, t_k with the following properties:

(1) $\liminf_{k\to\infty} |t_k| \geq 2$
(2) There exists an $\eta > 0$ so that $\frac{d\mu_{n_k}}{dt}(t_k) > \eta$ for a sequence of natural numbers, $n_k \uparrow \infty$.

By Stieltjes inversion formula, $\exists\ h_k > 0$ *s.t.* $\forall\ h \in (0, h_k)$ we have the following:

$$\frac{\Im(F_{\mu_{n_k}}(t_{n_k} + ih))}{|F_{\mu_{n_k}}(t_{n_k} + ih)|^2} = -\Im(G_{\mu_{n_k}}(t_{n_k} + ih)) > \pi\eta$$

which, coupled with Lemma 4.1, implies

$$\eta\pi C^2 \leq \eta\pi|F_{\mu_{n_k}}(t_k + ih)|^2 < \Im(F_{\mu_{n_k}}(t_k + ih)). \tag{1}$$

Recall that Lemma 2.5 implies that $F_{\mu_{n_k}}(t_k + ih) + \varphi_{\mu_{n_k}}(F_{\mu_{n_k}}(t_k + ih)) = t_k + ih$. This tells us that $\Im(F_{\mu_{n_k}}(t_k + ih)) + \Im(\varphi_{\mu_{n_k}}(F_{\mu_{n_k}}(t_k + ih))) = h$. Now, with $a \in \mathbb{R}$ as in Lemma 2.5 (3), we have that

$$\Im(\varphi_{\mu_{n_k}}(F_{\mu_{n_k}}(t_k + ih))) \leq |\varphi_{\mu_{n_k}}(F_{\mu_{n_k}}(t_k + ih)) - a| \leq \frac{2}{\Im(F_{\mu_{n_k}}(t_k + ih))}.$$

Furthermore, Lemma 2.3 tells us that $\Im(F_{\mu_{n_k}})(t_k + ih) > 0$ and we have the following:

$$\begin{aligned} h = \Im(F_{\mu_{n_k}}(t_k + ih)) + \Im(\varphi_{\mu_{n_k}}(F_{\mu_{n_k}}(t_k + ih))) &\geq \Im(F_{\mu_{n_k}}(t_k + ih)) - |\varphi_{\mu_{n_k}}(F_{\mu_{n_k}}(t_k + ih))| \\ &\geq \Im(F_{\mu_{n_k}}(t_k + ih)) - \frac{2}{\Im(F_{\mu_{n_k}}(t_k + ih))}. \end{aligned}$$

This implies that

$$\Im(F_{\mu_{n_k}}(t_k + ih)) \leq M \tag{2}$$

where M is a constant and h is sufficiently small, independent of t_k.

Thus, by (1) and (2) , $(F_{\mu_{n_k}}(t_k + ih))$ lies entirely in the truncated disk, $\Omega = \{z : |z| < K,\ \Im(z) > \eta\pi C^2\}$. By infinite divisibility, $\varphi_{\mu_{n_k}}$ is defined on $F_{\mu_{n_k}}(\mathbb{C}^+) \cap \Omega$ and, by Lemma 3.2, converges to z^{-1} uniformly on this set. However, $\{z + z^{-1} : z \in \Omega\}$ contains no neighborhood of $\mathbb{R}$ outside of $[-2+\delta, 2-\delta]$ for some fixed $\delta > 0$. The same must also hold for $z + \varphi_{\mu_{n_k}}(z)$ for n_k big enough. Therefore, $t_k + ih = F_{\mu_{n_k}}(t_k + ih) + \varphi_{\mu_{n_k}}(F_{\mu_{n_k}}(t_k + ih))$ must be contained in a neighborhood of $[-2+\delta, 2-\delta]$ in $\mathbb{C}$ of as small a size as we wish for h small enough and k big enough. But this contradicts the fact the t_k must eventually leave $[-2+\frac{\delta}{2}, 2-\frac{\delta}{2}]$. Thus, our theorem holds.

□

It would be interesting to see whether Lemma 4.1 or Theorem 4.2 holds without the assumption that μ is infinitely divisible.

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